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On growth walks with self-avoiding constraints

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Abstract. We study a general class of growth walks in two dimensions with a self-avoiding constraint, which can be considered as 'dressed' self-avoiding walks. A walker travels on a square lattice and chooses at each step one of the three forward directions with probability p_1, p_2 and p_3 if the directions are allowed. The self-avoiding constraint is obtained by 'dressing' the walk with 'black' and 'white' particles after each step such that the walk generates the external perimeter of arbitrary clusters consisting of black or white particles. We discuss the critical values of p_1, p_2 and p_3 which allow for infinite walks and study the end-to-end distance $r(t)$ and the corresponding distribution function $N(r, t)$. Furthermore, we introduce a bias field in the growth process and discuss how the time evolution of the walk is modified by the bias.

1. Introduction

In recent years, considerable attention has been addressed to kinetic growth models. Kinetic growth models have been used to describe a large variety of growth phenomena, ranging from the irreversible growth of additive copolymerisation to the growth of tumours, epidemics, diffusion fronts and signal propagation [1-11]. In this paper we will concentrate on those growth models which can be considered as self-avoiding random walks. Self-avoiding walks (SAW) were proposed by Hammersley and Morton [2] as a model for linear polymers with excluded volume interactions. Originally, the SAW was defined in equilibrium, where the ensemble of chains of fixed number N of elements is considered and crossing between the elements of one chain is forbidden. On average, the radius R and 'mass' N of a chain are related by

$$N \sim R^{d_F}$$

where d_F is the fractal dimension of the chain: for the SAW, $d_F = \frac{4}{3}$ in two dimensions. In principle, the SAW can also be considered as a dynamical process, as a normal random walk which stops when a site is going to be visited twice. However, in this way only short chains can be grown and large chains of N segments are exponentially rare.

In order to allow also for the growth of *larger* chains with a self-avoiding constraint, several other kinetic growth models have been introduced in recent years. In the kinetic growth walk (KGW), discussed by Majid *et al* [12], a random walker can only step to those sites which have not been visited before. The walk is terminated when the walker is trapped. However, in this walk chains of large N are still rare and the asymptotic behaviour of the walk is difficult to detect; for small N the KGW differs considerably from the SAW [4, 12, 13] but it turned out [13–15] that both walks nevertheless show the same (asymptotic) behaviour for large N with the same fractal dimension d_F.

More recently, random walks with a self-avoiding constraint which allow the growth of larger chains have been proposed by several authors. Ziff *et al* [16] have introduced a kind of two-sided self-avoiding walk. In this walk the external perimeter of percolation clusters in two-dimensional systems is generated, which by definition is self-avoiding. The walk stops only when the perimeter is formed, i.e. when the last added segment of the chain is linked with the initial segment. It has been shown by Weinrib and Trugman [17] that this perimeter walk shows the same asymptotic behaviour as two other (equivalent) kinetic growth models, the 'indefinitely growing self-avoiding walk' (IGSAW) [18] and the 'smart kinetic walk' (SKW) [17]. In the SKW, for example, a walker steps with equal probability to allowed nearest-neighbour sites. Allowed sites are those sites which have not been visited before and which are connected via allowed sites with infinity, so that the walker cannot be trapped.

In this paper we introduce and discuss a more general type of growth walk with a self-avoiding constraint, which can be considered 'dressed self-avoiding walks'. A walker travels on a two-dimensional lattice with coordination number z and chooses the direction of the next step with probability $p_1, p_2, \ldots, p_{z-1}$ if the directions are allowed (the backwards direction is always forbidden). The self-avoiding constraint is obtained by 'dressing' the walk with 'black' and 'white' particles after each step such that the walk generates the external perimeter of arbitrary clusters consisting of black or white particles. The perimeter walk model of Ziff *et al* [16] is a particular case of this growth model. Furthermore, we introduce a bias field in the growth process and discuss how the time evolution of the chain is modified by the field.

The paper is organised as follows. In § 2 the model is explained. In § 3 the critical behaviour of the walk as a function of the jump probabilities $p_1, p_2, \ldots, p_{z-1}$ is investigated and a scaling theory for the end-to-end distance $r(t)$ and the corresponding distribution function $N(r, t)$ is performed. In § 4 we study the effect of the bias field on the growth process and § 5 concludes the paper with a discussion.

2. The dressed self-avoiding walk (DSAW)

For convenience we consider a square lattice and its dual lattice ($z=4$). Adjacent sites are separated by bonds which form the dual lattice (see figure 1(a)). The walk is performed on the dual lattice and self-avoiding constraint will be obtained by 'dressing' the walk with 'black' and 'white' particles which are placed on the original lattice.

At time $t=1$ we occupy one bond in the dual lattice by an arrow which defines the direction of the walk. This arrow is the seed of our growth process. In order to make the growth walk self-avoiding (no crossing and no trapping) we dress the arrow by placing two different types of particles on both sides of the arrow on the orginal lattice, a 'white' particle on the left and a 'black' particle on the right (figure 1(a)). At time $t=2$ one of the three bonds in the forward direction of the arrow (left, straight ahead or right) is chosen with probabilities p_1, p_2 or p_3, respectively, and then occupied

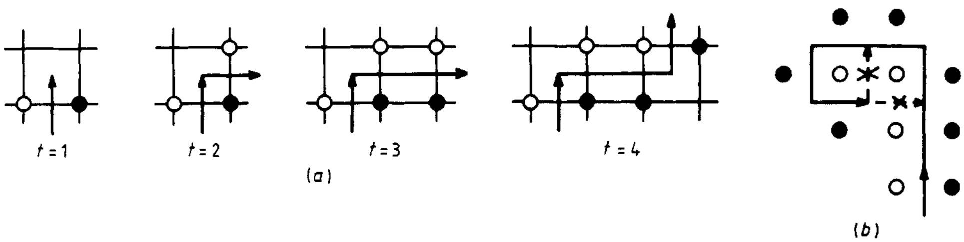
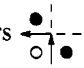
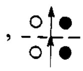
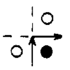
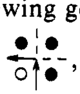
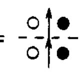
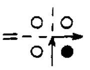


Figure 1. (*a*) Initial steps of a 'dressed' self-avoiding walk determined by the set of generators ($L'AR'$). (*b*) Pattern of a 'dressed' self-avoiding walk grown by the set of generators ($L'AR'$). In the next step a left turn and a move straight ahead are forbidden (broken arrows), and only a turn to the right is allowed.

by a second arrow. By definition, $p_1+p_2+p_3=1$. Then the new arrow is dressed by adding a black and/or white particle, such that the right side of the arrow is occupied by a black particle and the left side is occupied by a white particle (figure 1(*a*)). It is convenient to combine two successive steps of the walk and to consider them as the generators of the growth process (see also [19]). According to the three available directions, left, straight ahead, and right, we have the three generators ,  and . We shall see below that other choices of the generators are also possible.

The generators are chosen with probabilities p_1, p_2 and p_3 and the growth walk is created by successively adding generators. The general procedure is as follows. Suppose we are at time step t and want to proceed to $t+1$. First we choose the bond where we attempt to place the new arrow. This new arrow and the last added arrow define the generator. The growth proceeds in the attempted direction, if the generator fits into the existing environment of black and white particles. For example, if adding the generator requires us to place a black particle on a site which is already occupied by a white particle, then the attempted direction is not allowed, time is not increased, and a new trial must be made (see figure 1(*b*)). By this procedure, the self-avoiding constraint is always satisfied. The walk terminates when the last added arrow is linked together with the initial arrow and a closed loop is formed.

As indicated above, the generators are in general not uniquely defined. For the square lattice, one has the choice between the following generators to create a DSAW. For turning left, one can choose $L'=$  or $L''=$ , for moving ahead we have only $A=$  as above, and for turning right we can either choose $R'=$  or $R''=$ .

The walk defined by the set (L', A, R') has been discussed above. In this walk, black and white particles play a symmetric role. Different bonds can touch each other since there is an unoccupied corner in L' and R', but crossing is impossible.

When the set (L'', A, R'') is chosen, the symmetry between black and white is preserved. Now bonds cannot even touch each other and the walk is strictly self-avoiding.

Finally, the sets $S_1 \equiv (L'', A, R')$ and $S_2 \equiv (L', A, R'')$ of generators create DSAW where black and white particles play an asymmetrical role. For the special choice of $p_1 = p^2$, $p_2 = p(1-p)$ and $p_3 = (1-p)$, the set S_1 generates the perimeter of site-percolation clusters made of black particles (with concentration p) on the square lattice and the walk reduces to the perimeter walk of Ziff *et al* [16]. The set S_2 is equivalent to S_1 when black and white are permuted.

In general, the number of possible generators increases with increasing coordination number of the dual lattice. In a triangular lattice, for example, there is only one set of generators possible. The corresponding dual lattice is the honeycomb lattice ($z=3$) with only two forward directions, left and right.

Here we consider the square lattice only. As we expect that all walks described above belong to the same universality class, we will choose in the following only one specific set of generators, S_1 (figure 2). First we will study how the choice of p_1, p_2, p_3 influences the properties of the growth walk.

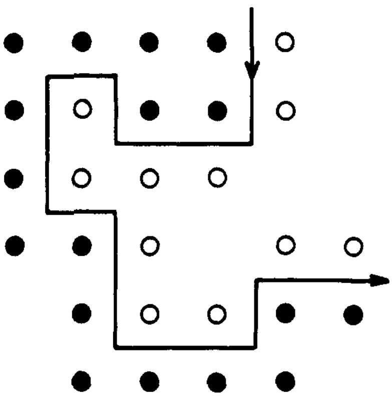


Figure 2. Pattern of a 'dressed' self-avoiding walk grown by the set of generators $(L''AR')$.

3. Scaling properties of the dressed self-avoiding walk on the square lattice

3.1. Critical behaviour

In the normal perimeter walk on the square lattice, p_1, p_2, p_3 are given by $p_1 = p^2$, $p_2 = p(1-p)$ and $p_3 = (1-p)$, where p is the (given) concentration of black particles [16]. At the critical value $p = p_c \simeq 0.5928$ the black particles start forming an infinite cluster [20]. Correspondingly, there exists an infinite cluster perimeter at p_c and an infinite perimeter walk can be generated.

In the general DSAW the growth properties depend on p_1, p_2 and p_3 with $p_1 + p_2 + p_3 = 1$. The walk is determined by two independent parameters and we therefore expect a critical line where an infinite walk can occur. More accurately, at the critical line the probability of finding an unclosed walk of length h follows a power law in h, while it decreases exponentially otherwise.

To determine this critical line we consider the distribution of closed loops and distinguish between left- and right-turning loops. In right-turning loops, the black particles are inside the loop and the walk generates the external perimeter of a finite black cluster. In left-turning loops, the black particles are outside the loop and the walk generates one of the internal perimeters of a black cluster. Below criticality, there exist more external than internal perimeters, while above criticality we have more internal than external perimeters. Only at criticality are the number of external and internal perimeters exactly the same [21]. By counting the number $n_h^{\rm ext}$ and $n_h^{\rm int}$ of right- and left-turning closed loops of h steps (arrows) we can determine the critical line. From the scaling relations [20] for ordinary percolation,

$$n_h^{\rm ext} = h^{-\tau_h} f[(p - p_c)h^{\sigma_h}] \tag{1a}$$

$$n_h^{\rm int} = h^{-\tau_h} f[(p_c - p)h^{\sigma_h}] \tag{1b}$$

we find that close to the critical point the ratio between $n_h^{\rm ext}$ and $n_h^{\rm int}$ is given by

$$n_h^{\rm ext}/n_h^{\rm int} \simeq 1 + 2(p - p_c)h^{\sigma_h} f'(0)/f(0) \tag{2}$$

where $\sigma_h = 1/(1+\nu)$ and $\nu = \frac{4}{3}$ is the correlation length exponent. Therefore, for large h this ratio is very sensitive to a variation in p near p_c, which allows a very accurate determination of the critical points. For convenience we have taken p_2 and p_1/p_3 as independent parameters: p_2 is related to the mean length between two turns of the walker and therefore defines a basic scale in the problem and p_1/p_3 is the ratio between left and right turns. The result for the critical line is shown in figure 3. When $p_2 = 0$, i.e. only left and right turns are allowed, then the critical point is given by $p_3 = p_c' = 0.5928$ and $p_1 = 1 - p_c$. The proof is sketched in appendix 2. On the other hand, when p_2 tends to unity, the walker greatly prefers to go straight ahead. In this case, the ratio between the critical values of p_1 and p_3 tends to unity. Theoretical arguments for this are also given in appendix 2.

3.2. *The time evolution of the end-to-end distance*

Next we discuss the evolution of the end-to-end distance $\langle r^2(t)\rangle^{1/2}$ as function of t on the critical line. By definition, t is equal to the number h of segments (arrows). A

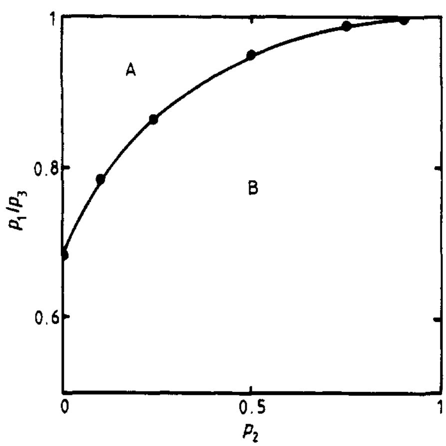


Figure 3. Critical line in the space of jump probabilities (p_1, p_2, p_3). The critical line separates the region above criticality (A), where preferentially internal perimeters of black clusters are generated, from the region below criticality (B) where preferentially external perimeters of black clusters are generated. The full circles are results from Monte Carlo simulation. For one set of parameters, walks up to 3200 time steps have been analysed and averages over 100 000 walks have been made.

basic length in the problem is provided by $a_{\text{eff}} = 1/(1-p_2)$, which is proportional to the mean length $\bar{l}$ and the mean time $\bar{t}$ between two turns, $\bar{l} = \bar{t}$. For small times we must have

$$\langle r^2(t)\rangle \sim t^2 \qquad t \ll \bar{t} \tag{3a}$$

while for large times we expect that the walks belong to the same universality class as the normal perimeter walk (see above), i.e.

$$\langle r^2(t)\rangle \sim t^{2/d_{\text{H}}} \qquad t \gg \bar{t} \tag{3b}$$

where $d_{\text{H}} = \frac{7}{4}$ is the fractal dimension of the external perimeter ('hull') in two dimensions. Since a_{eff} is the basic length *and* time in the problem we expect that for large enough a_{eff}, $\langle r^2\rangle$ and t can be expressed in units of a_{eff}, i.e.

$$\langle r^2(t)\rangle / a_{\text{eff}}^2 = R^2(t/a_{\text{eff}}) \tag{4}$$

as $\bar{l} \propto a_{\text{eff}}$ and $\bar{t} \propto a_{\text{eff}}$. The scaling function $R(x)$ is universal and should not depend on the walk parameters. From $(3a, b)$ and (4) we obtain

$$R(x) \sim x \qquad x \ll 1 \tag{5a}$$

$$R(x) \sim x^{1/d_{\text{H}}} \qquad x \gg 1. \tag{5b}$$

Combining (4) and $(5b)$ we can predict that for large t the mean end-to-end distance varies as

$$\langle r^2(t)\rangle^{1/2} \sim (a_{\text{eff}})^{1-1/d_{\text{H}}} t^{1/d_{\text{H}}}. \tag{6}$$

We have calculated $\langle r^2(t)\rangle$ for various values of a_{eff} on the critical line. We have scaled the result according to equation (4). The data collapse, shown in figure 4, confirms the scaling theory, and shows clearly that the asymptotic behaviour of the walk is described by the external perimeters of ordinary percolation clusters, as has been assumed in $(3b)$.

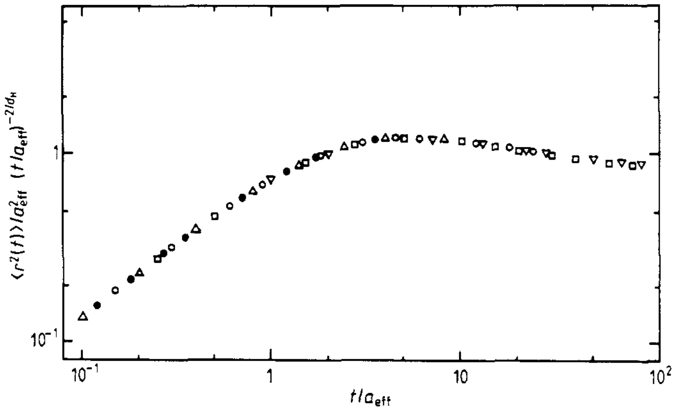


Figure 4. Plot of $\langle r^2(t)\rangle / a_{\text{eff}}^2 (t/a_{\text{eff}})^{-2/d_{\text{H}}}$, $d_{\text{H}} = \frac{7}{4}$, as a function of t/a_{eff} at criticality for several values of a_{eff} ($a_{\text{eff}} = 100$ (○), 50 (△), 33.3 (●), 20 (□), 10 (▽)). The results are from Monte Carlo simulations where, for each value of a_{eff}, averages over typically 50 000 walks have been made. The data collapse supports the scaling assumption, equation (4).

3.3. The distribution function $N(r, t)$ on the critical line

Next we study the distribution function $N(r, t)$ at criticality, which is defined as the probability of finding the walker at time t at distance r from the seed. For all critical walks with arbitrary value of $a_{\text{eff}} = 1/(1-p_2)$ and for large times we expect the following 'superscaling' behaviour:

$$N(r, t) = (1/\bar{r}) N_s(t) \mathcal{N}(r/\bar{r}) \tag{7}$$

where $N_s(t)$ is the number of surviving walks (which are not closed) at time t and $\bar{r} = \langle r^2(t, a_{\text{eff}}) \rangle^{1/2}$. From equation (6) we know that $\bar{r} \sim a_{\text{eff}}^{(1-1/d_{\text{H}})} t^{1/d_{\text{H}}}$ for $t \gg \bar{t}$ and $a_{\text{eff}} \gg 1$. For large t and fixed a_{eff}, $\bar{r}$ scales as $t^{1/d_{\text{H}}}$ and (7) takes the scaling form (see also [1, 8])

$$N(r, t) = N_s(t) t^{-1/d_{\text{H}}} \tilde{N}(r t^{-1/d_{\text{H}}}). \tag{8}$$

By definition, both scaling functions $\mathcal{N}(x)$ and $\tilde{N}(x)$ are normalised. To confirm the scaling we have investigated $N(r, t)$ for several values of a_{eff} on the critical line. In figure 5 we have scaled $N(r, t)$ according to equation (8) for $a_{\text{eff}} = 1.318$ and $a_{\text{eff}} = 4$.

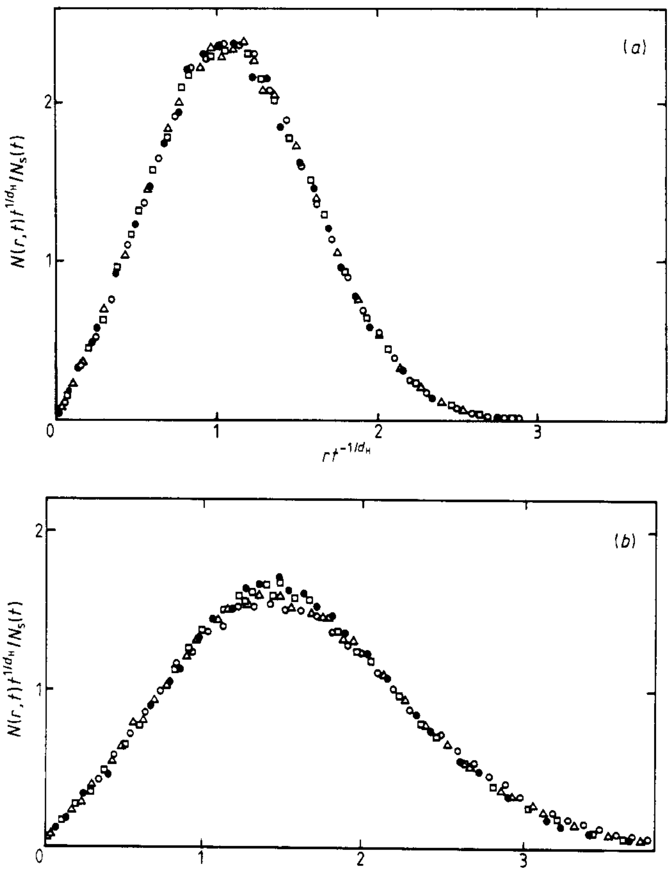


Figure 5. Plot of $N(r, t)t^{1/d_{\text{H}}}/N_s(t)$ as a function of $rt^{-1/d_{\text{H}}}$ at criticality for several values of t ($t = 400$ (○), 800 (△), 1600 (□), 3200 (●)) and for two values of a_{eff}. (a) $a_{\text{eff}} = 1.318$, which corresponds to the perimeter walk of Ziff *et al* [16]; (b) $a_{\text{eff}} = 4$. To obtain these results for each value of a_{eff} averages over 400 000 walks have been performed.

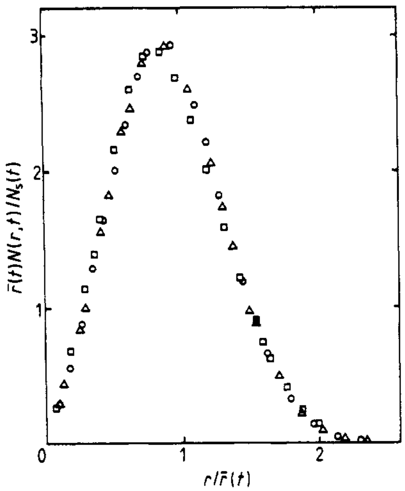


Figure 6. Plot of $\bar{r}(t)N(r,t)/N_s(t)$ as a function of $r/\bar{r}(t)$, ($\bar{r}(t) = (\langle r^2(t)\rangle)^{1/2}$) at criticality for several values of a_{eff}. The different symbols correspond to different values of a_{eff} ($a_{\text{eff}} = 1$ (○), 1.318 (△), 4 (□)). For each value of a_{eff} averages over typically 400 000 walks have been made. The data collapse supports the 'superscaling' assumption, equation (7).

The data collapse confirms the scaling expression. In figure 6 we have checked the 'superscaling' equation (7) for three values of a_{eff}. Again we obtain a data collapse which also shows that the 'superscaling' is valid.

4. The biased perimeter walk

After having discussed scaling properties of the dressed self-avoiding walk on the square lattice, we will now consider the influence of an external bias field on the time evolution of the growth process.

The bias field is introduced as follows. We assume that the bias field points in the y direction and is described by a parameter $\delta \geqslant 0$. The probability $P_{+y}(P_{-y})$ the walker steps in forward (backward) direction of the field is enhanced (decreased), while steps in the perpendicular directions P_{+x} and P_{-x} are not affected. Our choice for the bias is motivated by thermally activated diffusion in the presence of an external field E. For jumps in direction of the field, the potential barrier Δ is decreased, $\Delta \rightarrow \Delta - E$ while for jumps in the opposite direction, $\Delta \rightarrow \Delta + E$. Consequently we assume

$$P_{+y}(E) = P_{+y}(0)(1+\delta) \tag{9a}$$

$$P_{-y}(E) = P_{-y}(0)/(1+\delta) \tag{9b}$$

$$P_{\pm x}(E) = P_{\pm x}(0) \tag{9c}$$

where

$$1+\delta \equiv \exp(E/k_{\mathrm{B}}T) \qquad E \geqslant 0. \tag{9d}$$

A general biased DSAW (on a square lattice) is characterised by four quantities: p_1, p_2, p_3 and δ. At each time step, the probabilities p_1, p_2, p_3 are modified by the bias field according to (9a)-(9c) and the growth walk proceeds as described in § 2. We will study only one point in the critical line: $p_1 = p_c^2$, $p_2 = p_c(1-p_c)$, $p_3 = 1-p_c$. By this walk (without bias) the external perimeters of ordinary percolation clusters at p_c are generated. We have studied the end-to-end distance for several values of δ by Monte Carlo simulation. The results are shown in figure 7. For small bias fields $\delta \ll 1$ a well pronounced crossover time t_x occurs. For small times, $1 \ll t \ll t_x$, $\langle r^2(t)\rangle \sim t^{2/d_H}$ as in the unbiased case, while for large times $t \gg t_x$ we find the conventional result for biased diffusion in uniform media, $\langle r^2(t)\rangle \sim t^2$. In contrast to biased diffusion on random networks, our medium has no fixed dead ends into which the walker can be trapped by the bias. Thus our simulations do not show unusual long time relaxation effects and give $r \sim t$. The crossover time t_x decreases with increasing δ. In order to establish

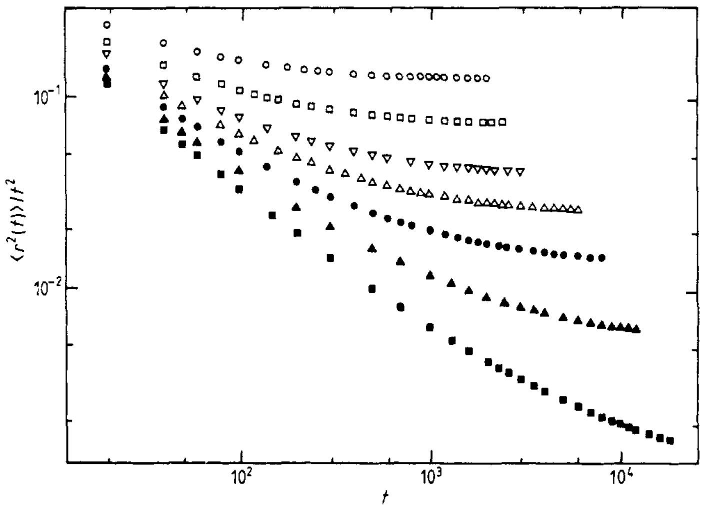


Figure 7. Plot of $\langle r^2(t)\rangle/t^2$ as a function of t for biased walks with $p_1 = p_c^2$, $p_2 = p_c(1-p_c)$, $p_3 = 1-p_c$ and several values of bias field strength δ ($\delta = 1$ (○), 0.7 (□), 0.5 (▽), 0.4 (△), 0.3 (●), 0.2 (▲), 0.1 (■)). For each value of δ we have averaged over typically 200 000 walks.

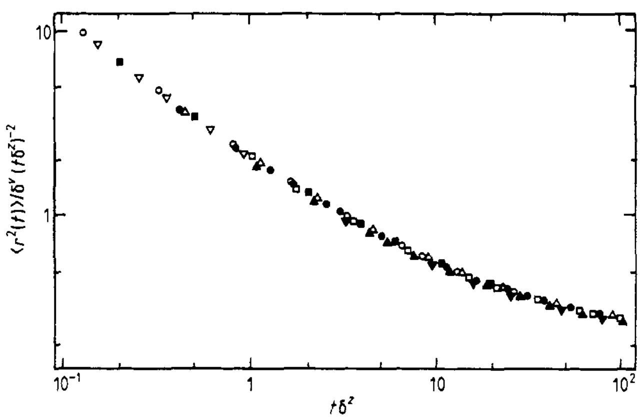


Figure 8. Plot of $\langle r^2(t)\rangle/\delta^{\nu}(t\delta^z)^{-2}$ as a function of $t\delta^z$ for biased walks with several values of δ ($\delta = 0.05$ (▽), 0.1 (○), 0.15 (■), 0.2 (△), 0.25 (●), 0.3 (□), 0.35 (▲), 0.4 (▼)). We have chosen $z = 2.8$ and for each value of δ we have averaged over typically 200 000 walks. The data collapse supports the scaling assumption, equation (10), and the choice of $z = 2.8 \pm 0.2$.

a scaling theory we make the ansatz

$$\langle r^2(t)\rangle = \delta^v B(t\delta^z) \tag{10}$$

with the new exponents v and z. For small times the walk is not affected by the small bias field and thus $\langle r^2(t)\rangle \sim t^{2/d_H}$ is independent of δ. Thus we require

$$B(x) \sim x^{2/d_H} \qquad x \ll 1 \tag{11}$$

and

$$v + 2z/d_H = 0 \tag{12}$$

which relates v and z to each other. To confirm the scaling ansatz and to determine the exponents v and z we have plotted $\langle r^2(t)\rangle/\delta^v(t\delta^z)^{-2}$ as a function of $t\delta^z$ for several values of z. The best fit was obtained for $z = 2.8 \pm 0.2$, which yields $v = -3.2 \pm 0.2$ (figure 8).

5. Conclusions

In this work we have examined a general class of self-avoiding walks where the self-avoiding constraint is introduced by 'dressing' the walks with two kinds of particles. A walker travels on a square lattice and chooses at each step one of the three forward directions with probability p_1, p_2 and p_3 if the directions are allowed. The self-avoiding constraint is obtained by 'dressing' the walk with 'black' and 'white' particles after each step such that the walk generates the external perimeter of arbitrary clusters consisting of black or white particles.

We have discussed the critical line in the space of the parameters p_1, p_2, p_3 where infinite walks occur, and studied the end-to-end distance r in the growth process as well as the corresponding distribution function $N(r, t)$.

We have also studied the growth walk in the presence of an external field δ with a fixed direction in space. In this case, the growth process as a function of time is characterised by a crossover time $t_x \sim \delta^{-z}$ which separates a fractal region with fractal dimensionality $d_H = \frac{7}{4}$ ($t \ll t_x$) from a one-dimensional region for $t \gg t_x$; we found for the crossover exponent $z \simeq 2.8 \pm 0.2$.

The general dressed self-avoiding walk can be considered as a growth model for polymers or for membranes in two-dimensional problems. The degrees of freedom in the growth process correspond to the probability that a new monomer is added in the different possible directions (three directions in the case of the square lattice). Furthermore, due to the various possibilities of dressing, the dressed self-avoiding walks contain additional degrees of freedom in their generation. The different types of dressing can be related to steric contraints in polymer growth or to hydrophilic and hydrophobic ends in membranes (black and white particles). Such characteristics seem to give very interesting features to dressed self-avoiding walks.

Acknowledgments

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Appendix 1. General structure of a 'dressed' self-avoiding walk

In this appendix we show that the concept of the DSAW is very general and can be applied to any two-dimensional structure. Consider an arbitrary two-dimensional lattice consisting of polygons (full lines in figure 9). The corresponding dual lattice is obtained by choosing one point inside each polygon and connecting the points in adjacent polygons (figure 9). The walk is performed on the dual lattice and the self-avoiding constraint will be obtained by dressing the walk with black and white particles on the sites of the original lattice. In figure 9, the walker's steps are presented by arrows on the bonds of the dual lattice and the growth process follows the description given in § 2. The self-avoiding constraints impose that the walker can never have particles of different colours (black and white) on the same side. He cannot walk again on his own path. A ring is formed when no other possibilities occur.

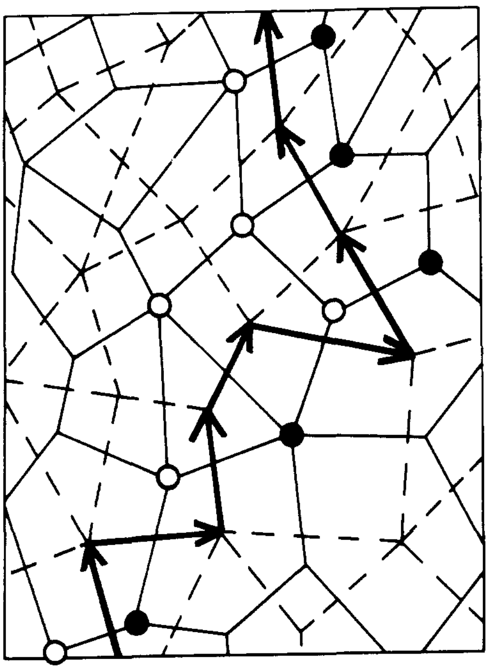

Figure 9. Graph of a general 'dressed' self-avoiding walk performed on an irregular two-dimensional lattice. The full lines represent the original lattice while the broken lines refer to the dual lattice. The steps of the walk are marked by arrows.

Appendix 2

(a) General features of the critical line. In § 3 we have shown that at criticality the number of right- and left-turning closed loops is the same. This is only possible if, on average, the walker turns to the right as often as to the left. Therefore, in the case of symmetric generators like $(L'AR')$ or $(L''AR'')$ the critical line is simply the line $p_1/p_3 = 1$. This is different from the case of § 3 where we have chosen the asymmetric generator $S_1 = (L''AR')$, which places two black particles on the lattice when turning to the left but only one white particle when turning to the right. Due to this asymmetry the ratio p_1/p_3 at criticality is lower than one, which can be understood by the following

qualitative argument. For $p_1/p_3=1$ the probability of turning to the right or to the left is equal in those cases where both directions are allowed. But due to the larger amount of black particles, the situation where the step to the right is forbidden occurs more often than the situation where the step to the left is forbidden. Therefore in total the walker turns to the left more often than to the right. To compensate this effect the ratio p_1/p_3 must be decreased to reach criticality.

As the difference in the amount of black and white particles depends on the number of steps to the right and to the left, this difference decreases with increasing value of p_2 and tends to zero for p_2 going to one, where the walker always steps straight ahead. Consequently, the ratio p_1/p_3 at criticality increases monotonically with increasing value of p_2 and tends to unity for p_2 going to one. This behaviour is obtained from our simulation results shown in figure 3.

(b) Critical behaviour at $p_2=0$. This is a particular feature of the walk generated by $S_1=(L''AR')$. When $p_2=0$, the walker turns at each step to the right or to the left. As can be seen from figure 10, this walk can be mapped on another walk which is performed on a $\sqrt{2}\times\sqrt{2}$ sublattice and can be considered as a simple perimeter walk of percolation clusters of white particles. In figure 10, the full arrows belong to the original walk $(p_1, 0, p_3)$ and the broken arrows to the corresponding walk on the $\sqrt{2}\times\sqrt{2}$ sublattice. In the sublattice walk the black particles on the extremity of the broken arrows are irrelevant. A white particle is placed on the sublattice when the original walk turns to the right and therefore occurs with probability p_3 while a (relevant) black particle is placed on the sublattice when the original walk turns to the left. Consequently this occurs with probability p_1. The generators of the sublattice walk, shown in figure 10(b), together with the way of occupying the sites of the sublattice, determine the sublattice walk to be the perimeter walk of Ziff *et al* [16], which generates the perimeter of percolation clusters of white particles. Therefore to reach criticality we have to put white particles on the lattice with probability p_c which yields $p_3=p_c$ and $p_1=1-p_c$. Consequently the ratio at criticality is given by

$$p_1/p_3=(1-p_c)/p_c \simeq 0.687$$

which is in excellent agreement with our results from Monte Carlo simulation, shown in figure 3.

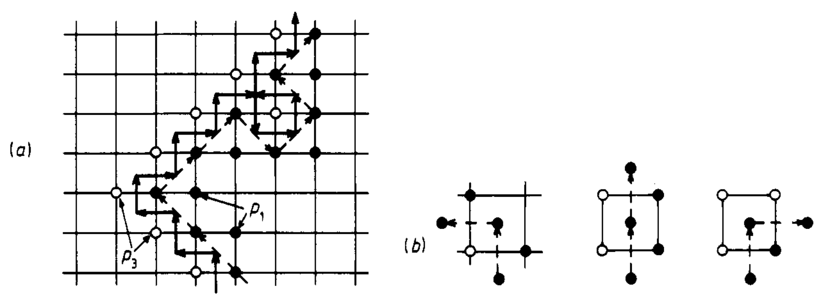

Figure 10. (a) Graph of a 'dressed' self-avoiding walk with $p_2=0$. The walk is marked by the full arrows. The broken arrows refer to the corresponding walk on the $\sqrt{2}\times\sqrt{2}$ sublattice. The generators of the sublattice walk are shown in figure 10(b).

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